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(First Draft)

According to Cantor, it can be shown by the diagonal method that there are infinitely more real than natural numbers since it is impossible to create a one-to-one correspondence (bijection) between them, and there are always real numbers without a natural number assigned to them. Obviously, the definition of the real and the definition of the natural numbers have a fundamental role in this case, as the definition prescribes their features–so new definitions of the natural numbers would change their nature. It is important to emphasize this, since the nature of natural numbers determined by the definition has fundamental role in the size of both the sets and the power sets. Cantor stated that the size of the latter was infinitely larger, but if we accept the definition of the natural numbers in its actual form, then Cantor's statement cannot be true. Besides examining this problem, we shall point out that a modified definition of the natural numbers makes possible a one-to-one correspondence between natural and real numbers, but unable to cause difference in the size of the sets and power sets.

On the Size of Natural Numbers

It is trivial that there is no "largest" natural number, as we can always find a larger one (e.g. adding 1 to the "largest one"). But the fact that a largest natural number does not exist, does not mean that a natural number can be infinitely large.

After all, if a natural number were infinitely large, then it would contain infinitely many digits. But according to the convention of "cardinal arithmetic", infinite+1=infinite; infinite + infinite = infinite and infinite x infinite = infinite:

$\aleph_0 + 1 = \aleph_0$,

$\aleph_0 + \aleph_0 = \aleph_0$,

$\aleph_0 \times \aleph_0 = \aleph_0$. (Stillwell 2010: 6)

But a natural number (that is, in this case, the number of the elements of the set) cannot be infinitely large (Tao 2006: 24): regardless of the size of the digit in a certain place in the number, the size of a natural number with infinitely many digits would be infinitely large. Thus the size of every number with infinitely many digits is infinitely large.

For example, 111…, 999… , 1212…, 20134… etc. would represent the same infinity. We would be then able to represent the same (infinitely large) number in infinitely many forms and because of it, according to today's mathematical tradition, there are no infinitely large natural numbers (notice that this kind of problem does not exist in the set of the real numbers: the size of 0.111… and that of 0.999… are evidently different).

The problem that there are no infinitely large natural numbers means that the well-known example of Hilbert's Hotel (Stillwell 2010: 4) is based on a misunderstanding: in this case, the first step is the declaration of the existence of "infinitely many" rooms and then we state that–according to the "cardinal arithmetic"–we can offer accommodation to infinitely many new guests although the hotel is full. Namely, $\aleph_0 + \aleph_0 = \aleph_0$. But there are no infinitely many rooms and guests since there are no infinitely many natural numbers.

Thus the example of Hilbert's Hotel illustrates only that, in the case of the set of the natural numbers, we always can add new rooms to the Hotel (as we always can add a number to a large natural number). It does not mean that there are infinitely many rooms or guests (since the size of their sets is represented by natural numbers), but we can solve the problem without referring to "infinitely large" natural numbers.

Of course, it is problematic that "infinity" is not a number in the sense that 0 or 137 or any other natural number is, if we accept the actual definition of natural numbers, since e.g. the rules of additions does not work in the case of infinitely large numbers (obviously, this is not a unique case: The rules of division do not work with 0). On the other hand: although it is problematic to handle infinity as an infinitely large number a mathematical system can be imagined where this problem does not appear–but it would require a reinterpretation of our mathematics.

On the Size of Power Sets

A power set is a given set's all subsets. If S = {A, B, C} then its power set is *S = {{A,B,C}, {A,B}, {A,C}, {B,C}, {A}, {B}, {C}, Ø}.

It is easy to see that if a set has n elements, then its power set is 2^n. Cantor–using the diagonal method–concluded from this that the power set of natural numbers has the same size as that real numbers'.

Obviously, the bijection between a set and its power set is not possible: If a set has more than 1 member then the power set has more members than the original set has (notice that the number of the elements of a set is the cardinality of the set and it is necessarily a natural number). But the fact that the size (cardinality) of a power set is larger than the original set's size, does not mean that the size of power sets is infinitely larger than the cardinality of the sets since this situation is fundamentally different form the relation of the natural and real numbers that are different types of numbers and their possible sizes are different (as mentioned earlier, there are only finitely many natural numbers and infinitely many real numbers).

But in this case both the elements of the sets and of the power sets are natural numbers, and if a natural number cannot be infinitely large (if there were an infinitely large natural number, then there would be infinitely many natural numbers, as well) then a power set where the number of elements is a natural number, cannot be infinitely large.

In short: the cardinality of a finite set is necessarily a natural number (and this natural number is the number of the set's elements), and since the number of both the sets' and power sets' elements are natural numbers, the size (cardinality) of the power sets cannot be infinitely larger than the size of the sets. After all, there are only finitely many natural numbers.

Thus while it is true that a certain set is smaller than its power set, it is false to say that there is no set larger than power set. If we have a power set that is 2^n, we can always construct a bigger set in the (2^n) +1 form. But neither the set nor the power set can be infinitely large, as both of their sizes are limited by the size of the natural numbers.

This means that–as opposed to today's mathematical opinion– the size of sets and the size of power sets are not different in the same way as the size of natural and real numbers. Both of them are only finitely large.

This is important not only because the size of power sets is not infinitely larger than the size of sets, but also because it means that we cannot construct an infinitely large power set that is based on a set, and because of that, we cannot construct an infinitely even larger set based on the infinitely large power set – and so on to infinity. Opposite to this, since the number of the elements of a power set is a natural number as well, this fact restricts their possible size, and because of this the size of the power set of a power set of a power set… (and so on) can be only finitely large, although there is no upper limit on its size.

Possible Solutions

Obviously, if we leave the definition of the natural numbers unchanged, we have to refuse Cantor's statement that the size of the power sets is infinitely larger than the size of the sets. If we would like to save the existence of infinitely larger power sets, then we have to change the definition of fhe natural numbers. On the other hand, this would lead to the reinterpretation of a part of the mathematics of the infinity and it would invalidate some results regarded to be valid today.

Perhaps it seems to be an acceptable solution to change the notion of the natural numbers to make possible the existence of infinitely large "natural numbers", containing infinitely many digits but they are equally infinitely large numbers. These "natural numbers" can be compared to each other regarding their digits to create an order (a "natural number" beginning with 123… would appear on the list before a "natural number" beginning with 223…).

Notice that this is not necessarily a problem: according to the actual conventions, there are numbers (what is more: there are infinitely many numbers) that are represented in two different forms, but their size is same: e.g. 0.999… = 1; 1.999… = 2 et cetera).

Of course, this is not the same as of the case of the natural numbers where–if we allow the existence of infinitely large “natural numbers”–infinitely many numbers would describe the same infinity.
If we allow the existence of infinitely large “natural numbers”, then we can create a one-to-one correspondence between the real and the “extended natural” numbers, for example:
123 → 0.123
1230 → 0.0123
and so on even in those cases where the real number contains infinitely many digits.
According to the actual interpretation, this bijection is impossible (since it would require bijection between real numbers and numbers with infinitely many digits), but this is not a problem if we accept the existence of infinitely large “natural numbers”.
But the existence of this kind of numbers does not offer a solution for the problem of the size of sets and power sets. According to the actual interpretation, there are infinitely less natural numbers than real numbers, because a natural number cannot be infinitely large. The same is true in the case of the size of the sets and power sets (namely, they cannot be infinitely large). If, however we reinterpret the meaning of the natural numbers, then it will be possible to accept the existence infinitely large “natural” numbers and, parallel to that, the existence of both infinitely large sets and power sets. But although a given power set would be larger than the set, there would not be a difference between the possible size of the sets and power sets: They would be infinitely large in the same way.
All in all, either we leave the meaning of the natural numbers unchanged or we change it, Cantor’s concept of the different infinites of the sets and power sets is not acceptable and similarly not acceptable the existence of an infinite tower of ascending infinities.
Thus the final question is whether it is possible to find a solution to cause the size of the sets and power sets different.

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