Divisibility of the amicable pairs, sociable chains, and the sums of the natural numbers and their proper divisors by ten

"The trouble with integers is that we have examined only the small ones. Maybe all the exciting stuff happens at really big numbers, ones we can't get our hands on or even begin to think about in any very definitive way." (Ronald Graham)

Abstract

According to our conjecture amicable pairs' fundamental feature that, in most cases, the sums of the pairs are divisible by ten. Similarly, the same seems to be true for sociable chains where the sums of the numbers of the whole chains in most cases are divisible by ten.
Extending the research to the natural numbers, it is probable that the larger the number, the more likely that the sum of the number and its proper divisors is divisible by ten. In short, this divisibility seems to be a fundamental feature of the natural numbers.

1. Amicable pairs and sociable chains

By definition, the proper divisor s(n) is the sum of the divisors of the number n including 1 but excluding the number itself. Amicable numbers (a,b) are those numbers where the sum of the proper divisors of n is equal to number b and vice versa: s(a) = b and s(b) = a. E.g. the smallest amicable pair is 220 and 284, where the proper divisors of 220 are 1, 2, 4, 5, 10, 11, 20, 22, 44, 55, 110 (=284) and the proper divisors of 284 are 1, 2, 4, 71, 142 (=220).
Sociable numbers' concept is a generalization of amicable pairs. Here the sum of the proper divisors of number a is equal to number b; the sums of the proper divisors of number b equal to number c… and so on to the last number of the chain where the sum of divisors is equal to number a.
In number theory the divisibility is a popular field of research, but not the divisibility of the sums of the numbers and their proper divisors.

2 Divisibility of sums by ten

2.1 Sum of the end digits of randomly chosen natural numbers

Choosing randomly two numbers, it is a 10 percentage probability hat the sums' last digit is 0 thus this sum is divisible by 10.In accordance with it, the ending digits of randomly chosen "pairs" where the sum is divisible by 10 in 499999 pairs (numbers are smaller than 1 million):

ending digits of pairs:	**how many:**
00	5112
19	5017
91	4992
28	4959
82	4852
37	5047
73	5023
46	4994
64	4998
55	5085
10.01%	50079

2.2 Sums of the amicable pairs divisible by 10

Opposite to the pairs of the randomly chosen natural numbers, “as the number of the amicable numbers approaches infinity, the percentage of the sums of the amicable pairs divisible by 10 approaches 100%”. To give an example, the sum of the 6232 and 6368 amicable pair is divisible by 10, thus the (6232, 6368) amicable pair belongs to the sequence (notice that the (220, 284) amicable pair does not belong to this sequence because its sum is 504).

Statistically, number of the pairs where the sums are divisible by 10:

first n pairs	how many	percentage
500	413	82.6
1000	825	82.5
1500	1264	84.26
2000	1705	85.25
2500	2150	86
3000	2593	86.43
3500	3047	87.05
4000	3495	87.375
4500	3956	87.91
5000	4406	88.12

What is more, it was examined 97,873 amicable number pairs where the smaller number is 30 digit long and where 97,01 % of the sums of these amicable numbers are divisible by 10. The examination of 258,155 amicable pairs with 210 digit length shows that only about 0.39 % (!) of the sums of the pairs are not divisible by 10. In these cases it only some more or less randomly chosen parts of the amicable pair sequences was examined, not from the first pairs–thus these percentages are higher than the real. But these data certainly show the trend (for details, see:
Sum of Amicable pairs conjecture: http://galantaiz.web.elte.hu/number-theory/Sum-of-amicable-pairs-conjecture.pdf).

2.3 Sums of the sociable chains divisible by 10

first n sociable chains	how many?	percentage
10	4	40%
100	77	77%
1000	949	94.9 %
1593	1532	96.17%

The biggest amicable chain has “only” 79 digit long numbers, but he growing percentage seems to be strengthen the conjecture that as the size of the numbers grows, there are more and more sociable chains where the sums of the numbers divisible by 10, and reaching infinity this reaches 100 %.

2.4 Sums of the natural numbers and their proper divisors

Interpreting a number and its proper divisors as a pair, the sums of the numbers and their divisors are more often divisible by 10 than the randomly chosen pairs’ sums:

first n numbers	how many	percentage
100,000	373709	37,3709
200,000	77449	38,7245
300,000	117887	39,29566666666667
400,000	158719	39,67975
500,000	199879	39,9758
600,000	241167	40,1945
700,000	282764	40,39485714285714
800,000	324455	40,556875
900,000	366279	40,69766666666667

1,000,000	408270	40,8270
1,100,000	450317	40,93790909090909
1,200,000	492460	41,03833333333333
1,300,000	534756	41,13507692307692
1,400,000	576699	41,19278571428571
1,500,000	619435	41,29566666666667
1,600,000	661900	41,36875
1,700,000	704424	41,43670588235294
1,800,000	747022	41,50122222222222
1,900,000	789701	41,56321052631579
2,000,000	832416	41,6208
2,100,000	875163	41,67442857142857
2,200,000	917947	41,72486363636364
2,300,000	960802	41,774
2,400,000	1003675	41,81979166666667
2,500,000	1046686	41,86744
2,600,000	1089649	41,90957692307692
2,700,000	1132521	41,94522222222222
2,800,000	1175579	41,98496428571429
2,900,000	1218672	42,0231724137931
3,000,000	1261822	42,06073333333333

For comparison: in the ad hoc chosen 10,000,000–10,100,000 interval this percentage is 44.163 %– between 2,900,000 and 3,000,000 it is only 43.147 %.

3 Conclusion

According to the y=1.2525ln(x) + 37,801 equation (based on the first 3 million numbers), somewhere between 1e+26 and 3e+26 almost all natural numbers with their sums are divisible by 10. It seems to be a plausible conjecture that as the number of the natural numbers approaches infinity, the percentage of the sums of the natural numbers and their divisors divisible by ten approaches 100%.

But in the case of the amicable numbers up to the first 5001 pairs, they are significantly smaller than 1e+26: the largest pair is 305,854,772,410 and 317,919,617,990 with only 12 digits. But up to this size, more than of 88% of amicable pairs meet the divisibility expectation although the equation predicts only about 55%.

The equation is probably not very accurate, but probably this is not the case behind this phenomenon.

The divisibility of the sum of the number and its proper divisors is only a necessary but not sufficient condition in the case of amicable and sociable numbers. There are enormously more numbers where the divisibility by 10 rule works than amicable or sociable numbers: only the percentage of those amicable or sociable numbers are higher where the divisibility by 10 rule is valid. Thus it seems reasonable to assume that there is a still unknown mathematical mechanism that chooses with higher probability those amicable and sociable numbers where the sum of the pairs or chains are divisible by 10.

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