Zoltán Galántai: Sum of Divisors Conjecture

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Summary

The aim of this short draft is to introduce a new conjecture stating that every positive integer larger than 8 can be constructed as a sum of the sums of two other numbers' proper divisors. This conjecture is similar to Goldbach's conjectures.

keywords*: Goldbach Conjectures, sum of proper divisors, building blocks, positive integers*

Definitions and Goldbach Conjectures

The Goldbach conjectures have two forms.

The weak (ternary) form states that every positive odd integer > 5 can be constructed as a sum of three primes (it is allowed to use more than once the same prime in the same sum). The weak Goldbach conjecture was proved by Helfgott (Helfgott: H. A.: The ternary Goldbach conjecture is true, 2013).

The strong (binary) form states that every positive even integer > 4 is a sum of two primes. Euler originally stated that every even integer >= 2 is a sum of two primes, but today 1 is not accepted as a prime number. Notice that erasing the first numbers from the list of the building blocks (that is 1 in this case) does not change the results in that sense that only the "threshold number" will be larger: erasing 1 from the list of building blocks means that the first integer constructed as the sums of primes will be 4 instead of 2. But, although we omitted 1, every integer larger than the threshold number will satisfy the number constructing criterions. So the Goldbach conjectures states that there are enough building blocks to construct every integer larger than the threshold number.

Lucky Numbers: A Model

Stanislaw Ulam and his colleagues introduced the concept of lucky numbers. Their method is based on a modified form of the sieve of Eratosthenes: They deleted every second number from the list of the positive integers; then they deleted every remained third… etc. The remaining list is: 1, 3, 7, 9, 13, 15, 21, 25, 31, 33, 37, 43, 49, 51, 63, 67, 69, 73, 75, 79, 87, 93, 99. . . (OEIS A000959). The list items (=building blocks) are the lucky numbers.

Ulam's team pointed out not only that the gap between two lucky numbers is roughly equal to the gap between two primes. They showed out that every even number up to 100,000 can be built as the sum of two lucky numbers. Walter Schneider showed that the weaker Goldbach-conjecture is true for this subset of numbers at least up to 10^10 (See e.g. David Wells: Prime Numbers. John Wiley and Sons, 2005).

On the Building Blocks

Although It is trivial, it is worth mentioning that the building blocks are by definition those numbers that are used to build a number. In the case of the Goldbach conjectures they are the prime numbers.

From a historical point of view, it is not a necessity to choose primes as the building blocks of positive integers. Alternatively, we could use, for example, the sums of proper divisors – although they are more complicated than the primes since the results based both on divisions and additions (while in the case of primes the building blocks are based only on divisions). Obviously, the simplest building block is the number 1 (it is not surprising that Aristotle regarded number on not as a "number" but as a "unity" of numbers). Choosing 1 we have to use only subtraction to analyze a number and to determine how many building blocks are needed to construct a number.

Analyzing a number to find its building blocks:

building block	subtraction	division	addition	multiplication
1	+	-	-	-
prime number	-	+	-	-
s(n) (sum of divisors)	-	+	+	-

Sum of Divisors Conjecture

Following Ulam's logics, we can apply the approach of the Goldbach conjectures for the sums of the proper divisors of the positive integers where s(n) is the sum of the proper divisors including 1 but not including the given number itself. E.g. if n=6 then s(n)=6 (since its divisors are 1,2, and 3). In other words: the building blocks are the sums of proper divisors.

Our conjecture is that

any number larger than 8 can be constructed as a sum of two sums of proper divisors of positive integers.

Some refinements to fine tuning this conjecture:

- There are building blocks on the list of the sums of the proper divisors that equal to integers (e.g. s(n) of 20 is 22), but this conjecture is about the possibility of constructing a positive integer as a sum of *two* building blocks, so we exclude this solution.
- Similarly, we use only those numbers' divisors' sums that are smaller than the given number. E.g. the sum of the divisors of 21 is 11, but 11 is not regarded as a building block of 21.

Notice that our above mentioned conjecture is analog to the strong form of Goldbach conjecture in that sense that it uses only two building blocks to construct appositive integer larger than 8.
This conjecture seems to be true

- even if we omit 1 as a building block and
- even if we prohibit the use of the duplication of building blocks (so 22 cannot be generated as 11+11 or 21+1, only as the sum of 15+7).

Appendix 1:

Sum of proper divisors between 1 and 100:

0, 1, 1, 3, 1, 6, 1, 7, 4, 8, 1, 16, 1, 10, 9, 15, 1, 21, 1, 22, 11, 14, 1, 36, 6, 16, 13, 28, 1, 42, 1, 31, 15, 20, 13, 55, 1, 22, 17, 50, 1, 54, 1, 40, 33, 26, 1, 76, 8, 43, 21, 46, 1, 66, 17, 64, 23, 32, 1, 108, 1, 34, 41, 63, 19, 78, 1, 58, 27, 74, 1, 123, 1, 40, 49, 64, 19, 90, 1, 106, 40, 44, 1, 140, 23, 46, 33, 92, 1, 144, 21, 76, 35, 50, 25, 156, 1, 73, 57, 117

Appendix 2:

Numbers constructed as the sums of two building blocks (sums of proper divisors) between 9 and 100:

1 ----
2 ----
3=1+1+1
4=3+1

5=3+1+1
6=3+3
7=6+1
8=6+1+1

9=6+3
10=7+3
11=8+3
12=8+4
13=7+6
14=7+7
15=8+7
16=10+6
17=8+9
18=15+3
19=10+9
20=16+4
21=15+6
22=15+7
23=15+8
24=16+8
25=15+10
26=16+10
27=21+6
28=22+6
29=16+13
30=14+16
31=15+16
32=21+11
33=22+11
34=28+6
35=28+7
36=28+8
37=31+6
38=31+7
39=28+11
40=20+20
41=28+13
42=20+22
43=40+3
44=16+28
45=42+3
46=36+10
47=36+11
48=40+8
49=33+16
50=43+7
51=42+9
52=42+10
53=43+10
54=456+8

55=22+33
56=43+13
57=54+3
58=55+3
59=26+33
60=50+10
61=50+11
62=40+22
63=50+13
64=42+22
65=50+15
66=50+16
67=50+17
68=55+13
69=66+3
70=63+7
71=63+8
72=64+8
73=64+9
74=64+10
75=64+11
76=66+10
77=66+11
78=63+15
79=64+15
80=74+6
81=74+7
82=74+8
83=50+33
84=78+6
85=78+7
86=50+36
87=32+55
88=33+55
89=40+49
90=50+40
91=42+49
92=43+49
93=90+3
94=90+4
95=55+40
96=76+20
97=76+21
98=92+6
99=92+7
100=92+8
etc.