Zoltán Galántai: Sum of amicable pairs conjecture

“Persons who have concerned themselves with talismans affirm that the amicable numbers of 220 and 284 have an influence to establish a union or close friendship between two individuals” (Ibn Khaldun)

"The search methods for amicable numbers are heavily based on primality testing of certain types of integers" (Song Y. Yan: Perfect, Amicable and Sociable Numbers. A Computational Approach)

Abstract
The aim of this short paper is to examine some features of the sums of the amicable pairs to point out that they are very likely to be divisible by 10. Some numerical results were published in the OEIS by me [1] and now some comments is added to them.

Introduction: why it is worth to mention
Amicable numbers (*a,b*) are those numbers where the sum of the proper divisors of *a* is equal to *b* and vice versa. E.g. the smallest amicable pair is 220 and 284, where the proper divisors of 220 are 1, 2, 4, 5, 10, 11, 20, 22, 44, 55, 110 (=284) and the proper divisors of 284 are 1, 2, 4, 71, 142 (=220). There are 1,220,455,470 known amicable pairs today [2].

Amicable pairs have been studying for more than 2 thousand years, but until now the research focused mainly on the separate features of them (e.g. for their proper divisors): To give an example, Gardner's conjecture on divisibility of sum of even amicable pairs (1968) stated that the sum of every amicable pair of even integers must be divisible by 9. It is known today that it is a false conjecture [3]. But it is more important from our point of view that we can study the features of number pairs instead of the individual numbers’ features. This was not a typical approach earlier – perhaps because we focused on the divisors of the amicable pairs. After all, they are per definitionem determined by the sums of the proper divisors of their individual numbers. Furthermore, some characteristics of the amicable pairs are not manifest if we examine only the first few, smaller pairs, but ignoring the bigger ones is a bit like saying after the examination of the prime numbers smaller than 10 that a significant part of the prime numbers are divisible by 2 while we do not know even all the basic properties of them.
Regardless of what the reason is, the features of the amicable pairs as not individual numbers were mainly ignored for a long time although it is an unanswered problem whether an even-odd pair of amicable numbers exists.

Sum of amicable pairs conjecture
According to our observations, “as the number of the amicable numbers approaches infinity, the percentage of the sums of the amicable pairs divisible by ten approaches 100%”. To give an example, the sum of the 6232 and 6368 amicable pair is divisible by ten, thus the (6232, 6368) amicable pair belongs to the sequence. But the (220, 284) amicable pair does not belong to this sequence because its sum is 504 [1].

Some numerical results
For the first 10 pairs, it is not manifest that the sums of the amicable pairs usually divisible by 10 since it is true only in 4 cases:

a	b	sum	divisible by 10?
220	284	504	not
1184	1210	2394	not
2620	2924	5544	not
5020	5564	10584	not
6232	6368	12600	**yes**
10744	10856	21600	**yes**
12285	14595	26880	**yes**
17296	18416	35712	not
63020	76084	139104	not
66928	66992	133920	**yes**

It is 40%, but the last digit of the sums in 50 % of the cases is 4 (see the underlined "nots" above). So the 40% seems not to be relevant at the first sight.
But the situation will change if you use a longer sequence. In our case the first 5,001 amicable pairs [4] were examined asking whether there is a regularity between the last digits of the numbers of the amicable pairs and whether their sums has a special divisibility feature.
The results show that the percentage of amicable pairs where the sum is divisible by 10 is increasing:

first n pairs:	percentage:
10	40
100	73
1000	82.5
2000	85.5
5001	88.1

Some 30 digits long amicable pairs

What is more, it was examined 97,873 amicable number pairs [5] where the smaller number is 30 digit long[i]. 97,01 % of the sums of these amicable numbers are divisible by 10.

Some 100 digits long amicable pairs

Performing the same examination for 100 digit long amicable pairs where the set of the studied numbers contained 516,211 amicable pairs[ii], the result seems to be not to refute the conjecture. In the case of the 30 digit long amicable pairs, a slightly more pairs' sums were not divisible by 10 (2.298783%) than in the case of the 100 digit long amicable pairs (2.87849%). It is about a 0,1 % difference.

Some 210 digits long amicable pairs

In accordance with the conjecture, the examination of 258155 amicable pairs[iii] with 210 digit length shows that only about 0.39 % are exceptions to the conjecture.
Notice that in these cases we examined only for some more or less randomly chosen parts of the amicable pair sequences, not from the first pairs–thus these percentages are higher than the real. But they certainly show surely well the trend.

What about the distributions of the ending digits?

Regarding the last digits of these amicable pairs, it seems to be noticeable for the first sight (in the case of the first 5001 pairs) that taking into account those combinations where sums of the last digits are 10 (1,9 and 91; 2,8 and 8,2 etc. treated as the same categories), the 5,5 combination takes about the 40 % percentage of the results and it takes almost 51 percentage with those pairs where the last

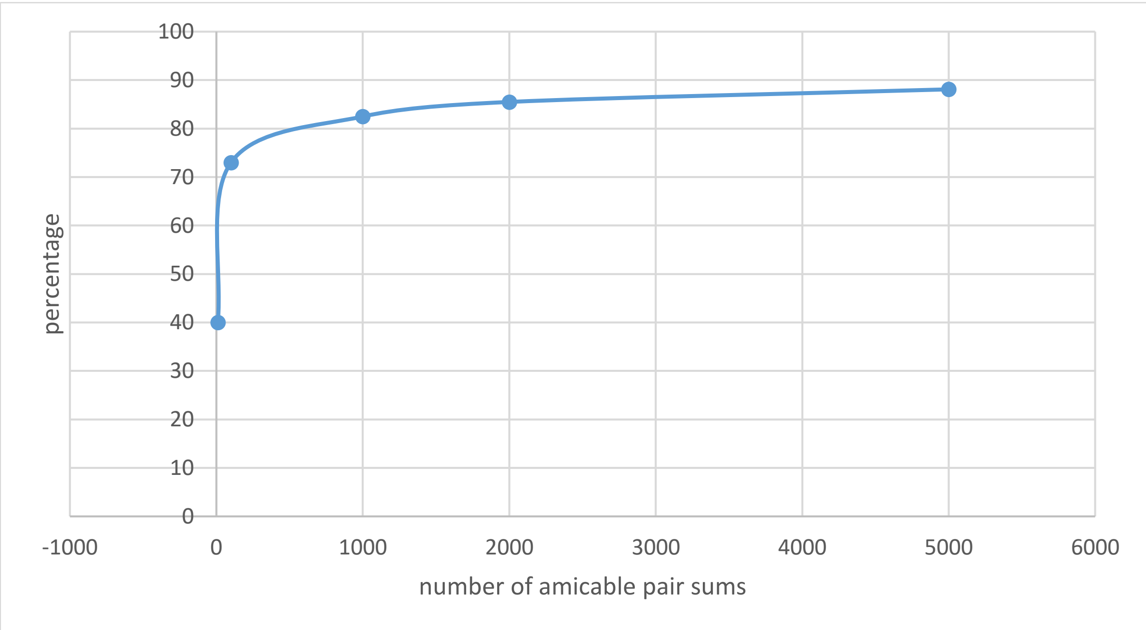
digits are zeros. But in the case of the examined part of the 210 digit long amicable pairs it is only 39.57 %. So it is plausible that their proportion gradually decreases and perhaps–but only perhaps– the proportions of the different endings where the sum is 10 become balanced if the size of the amicable pairs increases.
This observation draws the attention to the problem that if the most amicable pairs where the sum is divisible by 10 are divisible by 5 then this strange feature (sums' divisibility by 10) is intertwined with the divisibility of the individual amicable numbers: In short, there is a direct connection between the divisibility by 10 and the divisibility of the individual amicable numbers. But the situation is totally different if the divisibility by 10 is a result of an amicable pair where one last characters are 3 and 7. Namely, an amicable number where either 3 or 7 is the last digit cannot guarantee that it is divisible by either 3 or 7. The same is true for either an 1 and 9 last digit combination or if the last digits are 4 and 6 or 2 and 8: except for 2, the last digit does not guarantee the divisibility of the given amicable number by the given number.

Conclusion

In fact, those amicable pairs are seems to be exceptions whose sums are not divisible by 10–they in a certain sense are similar to the primes (since they aren't divisible any other number than 1 and itself, they are similarly rare). So it would be worth studying their features.

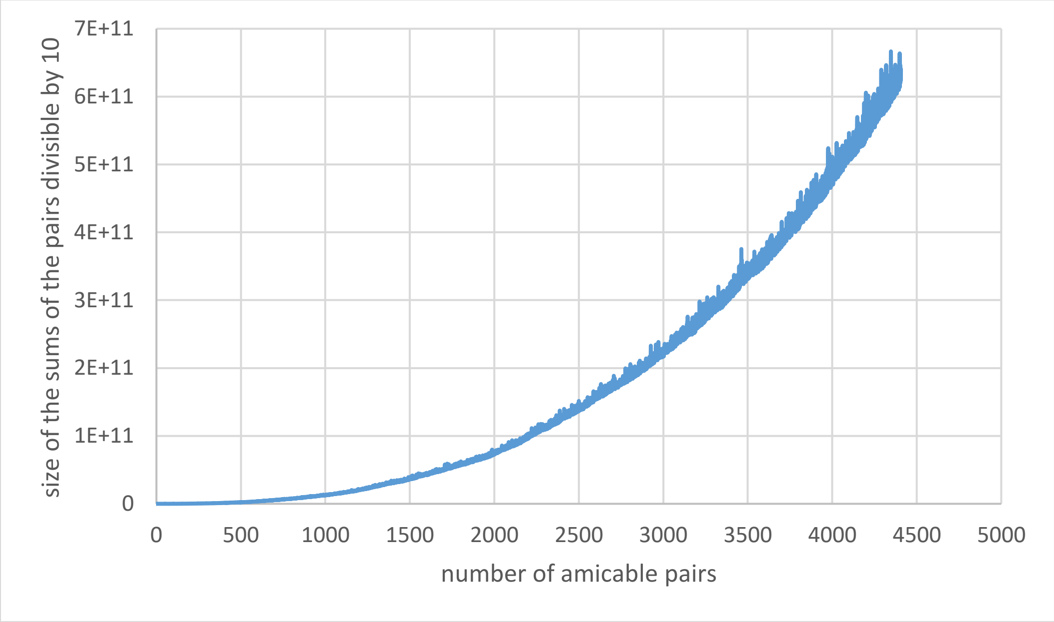
Appendix A

Graph: Percentage of amicable pairs where the sums of the pairs are divisible by 10:



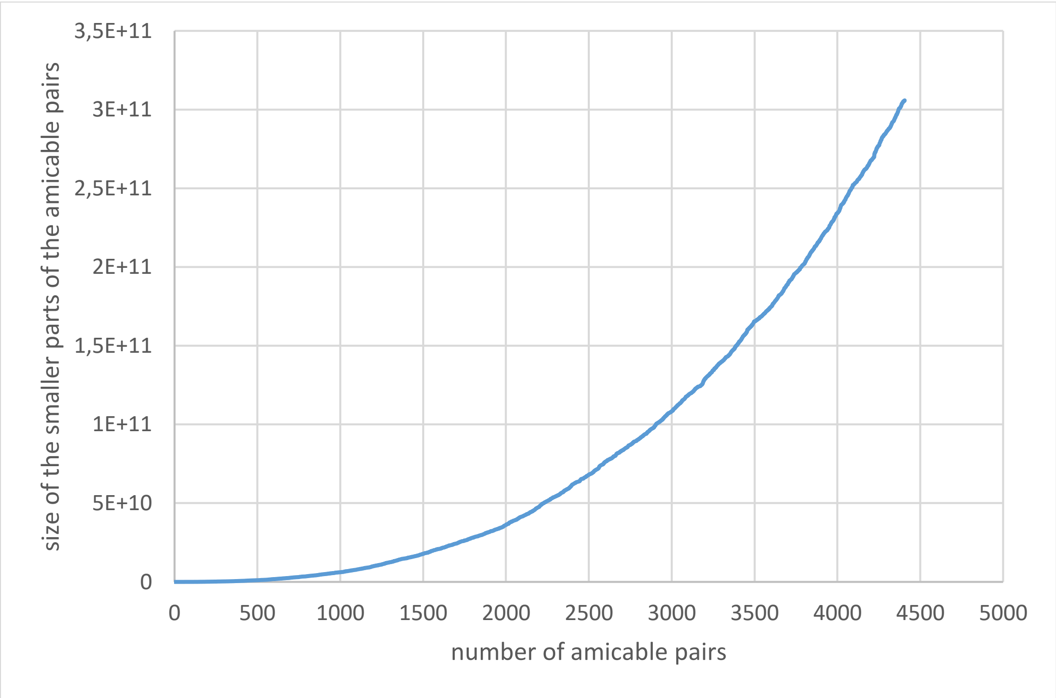
Appendix B

Graph: size of the sums of those amicable pairs where the sum is divisible by 10:



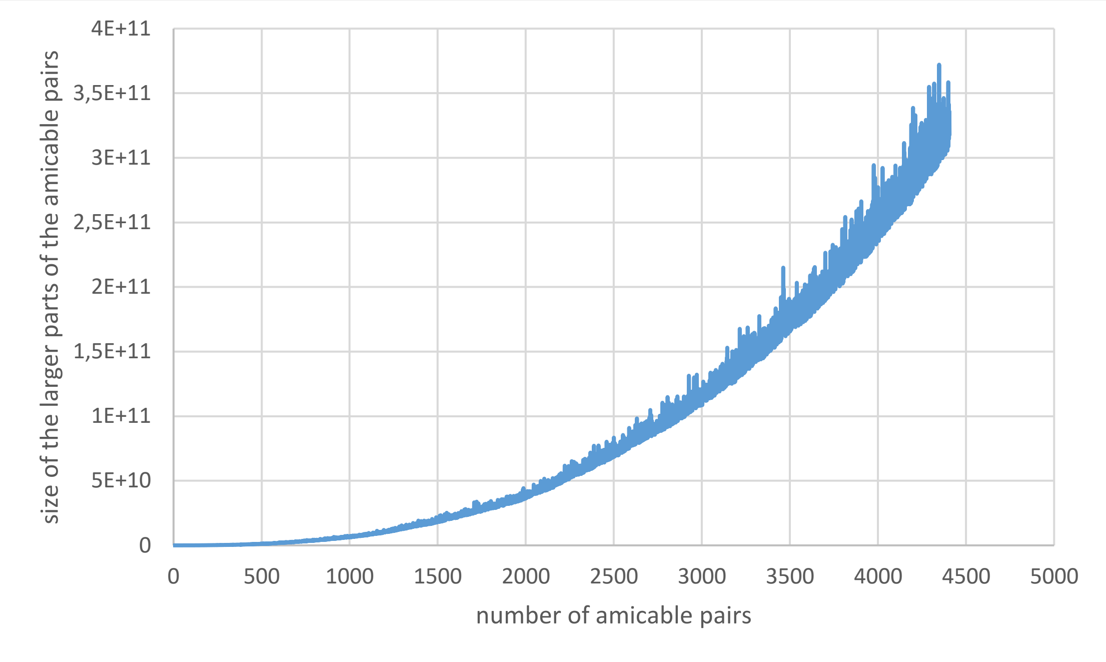
Appendix C

Graph: size of the smaller part of amicable pairs where the sum of the pairs is divisible by 10:



Appendix D

Graph: size of the larger part of amicable pairs where the sum of the pairs is divisible by 10:



Sources

[1] https://oeis.org/A291422
[2] https://sech.me/ap/
[3] https://proofwiki.org/wiki/Gardner%27s_Conjecture_on_Divisibility_of_Sum_of_Even_Amicable_Pairs
[4] http://djm.cc/amicable2.txt
[5] https://sech.me/ap/

Links:

List of the first 5001 amicable pairs with their sums denoting whether the sum is divisible by ten or not
List of the amicable pairs where the sum divisible by ten smaller and larger amicable numbers; sums (the first 4406 pairs)

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[i] the smallest pair of this sequence is
1000044614507379402672206462 50 and 1000592726559602138585419137 50
the biggest pair is
99999643509132755569242460553 7 and 10075618677271637724031657944 63

[ii] the smallest pair is:
18000001667986548190798358398352309734849138194258743572023487186985697121076326 5
9470198415945100676 and
18040099538933132026729218930368675728029655064754844499394149408371924900262028 5
3056727309750899324
the largest pair is:
19999998867007027806762936040312547699575301918471552778118925252653218382449236 7
9207104169128028472

20049879111667829847152695886936502153318593865313444903226839777103140033171483959579536
36224931528

[iii] smallest pair:
11000005574299003112695489928398493030695522748868154151267879856571104624795907792709664949042317396482992040646649152898230008983702850230882309344364295809825159081393883744895968557718148117633536 and
11001564046087036747774220410664252343947712225472986826424336230989692240266248633454476370752548572941904777255937505127980452983601982474726682444473093617177829821368415069814368271852766595966464

largest pair:
11999999713438786823000436374475416784257690698596321680458546845665759344252778849869303675119993220210365157271006878232548076031149877436939880936289527723193752265233648592905518707443586334226368 and
12169013796969053317639123019860913101196920728147526829074236740005348317543382880398112629396562984280975138700199082493434125314592463750318843631272204911426567237429947764723140056627362185709632