OBSERVATIONS ABOUT FIBONACCI SEQUENCES AND LUCAS SERIES: SOME NEW CONSTANTS

INTRODUCTION

This short paper is to present some connections between the Fibonacci numbers; and between the Fibonacci sequences and the Lucas series. Experimental mathematical methods were used. It is not our aim to exhaust this subject exploring every possible consequence thus we present only some examples to demonstrate that this approach can produce interesting results.
In parallel, some new constants will be introduced.

NOTATION

Treating the Fibonacci sequences, and the Lucas series as known,
the approximate value of phi = 1 + (sqrt 5)/2 = 1.618033988749895 (golden ratio). Obviously, values are only approximations in this paper.
Notation:
Fs : Fibonacci sequence.
Fs-1: instead of the nth item of the Fibonacci sequence, this is the n-1st
F1, F2... Fn are the items of the Fibonacci sequence in ascending order (Fn < Fn+1). Thus, Fn is the nth member of the list.
L1, L2... Ln are the items of the Lucas series following the above logic.

1. FIBONACCI PER FIBONACCI CONSTANTS:

It is known that the divisors of F2/F1; F3/F2... converge to the golden ratio. We can expand this method using a notation where
n = 0: dividing F1/F1; F2/F2; F3/F3...
n = 1: dividing F2/F1; F3/F2; F4/F3... (F2 = F1+1 = F1+n where n = 1)
n = 2: dividing F3/F1; F4/F2; F5/F3...
etc.
Results:

n	***constant***	***ph**n***		***Fs-1***		***Fs***		***phi***
0	1.0	= phi**0						
1	1.618033988749895	= phi**1	**OR**	0	+	1	*	phi
2	2.618033988749895	= phi**2	**OR**	1	+	1	*	phi
3	4.23606797749979	= phi**3	**OR**	1	+	2	*	phi
4	6.854101966249685	= phi**4	**OR**	2	+	3	*	phi
5	11.090169943749475	= phi**5	**OR**	3	+	5	*	phi
6	17.94427190999916	= phi**6	**OR**	5	+	8	*	phi
7	29.034441853748632	= phi**7	**OR**	8	+	13	*	phi
8	46.97871376374779	= phi**8	**OR**	13	+	21	*	phi
9	76.01315561749642	= phi**9	**OR**	21	+	34	*	phi

...
Some notes:
These results converge to certain constants just as the divisors of F2/F1; F3/F2... converge to 1.618033988749895 (golden ratio).
These constants can be written in two, different forms (since it is known that phi**n=(n-1)+n*phi).
Thus, in the second formula (nth constant=(n-1)+n*phi), the multiplicator numbers (1,1,2...) are equal to the Fs, while the numbers added to the formula are equal to the items of the Fs-1 list.
But there is one exception: n=0 does not fit into the pattern of this second formula. This n=0 can be written in the form of 0 + 0.61803398875 * phi (where 0.61803398875 is 1/phi) but, obviously,

0.61803398875 is not a part of the Fibonacci sequence. And, similarly, it is problematic that the value of Fs-1 is 0, since this does not fit into the Fs-1 pattern. On the other hand, this form manifestly shows not only the role of the phi, but the role of the Fibonacci sequence for the other values of n (n>0) as well.
Last, but not least, notice that the Lucas per Lucas constants are the same (that is not a surprise since the Lucas series are governed by the same rule as the Fibonacci sequences and only their starting values are different).

2. Lucas per Fibonacci constant:
As it is known, dividing L1/F1; L2/F2; L3/F3 etc., as the number of divisions approach infinity, the results of the divisions approach 2.23606797749979. So this constant is square root of 5.

3. Generalization of the Lucas per Fibonacci constant:
If instead of L1, we can divide L2/ F1; L3 with F2; L4 with F3... then this process results in 3.618033988749895 as a constant.
Generalizing this, if the first Lucas number divided by F1 is Ln, and the constants for n = 1,2,3...:

n	***constant***		***Fs***		***sqrt***	***Fs-1***		***2 - (1/phi)***
0	2.23606797749979		(1	*	sqrt 5) +	0	*	(2 - (1/phi))
1	3.618033988749895	=	(1	*	sqrt 5) +	1	*	(2 - (1/phi))
2	5.854101966249685	=	(2	*	sqrt 5) +	1	*	(2 - (1/phi))
3	9.47213595499958	=	(3	*	sqrt 5) +	2	*	(2 - (1/phi))
4	15.326237921249264	=	(5	*	sqrt 5) +	3	*	(2 - (1/phi))
5	24.798373876248842	=	(8	*	sqrt 5) +	5	*	(2 - (1/phi))
6	40.124611797498105	=	(13	*	sqrt 5) +	8	*	(2 - (1/phi))
7	64.92298567374695	=	(21	*	sqrt 5) +	13	*	(2 - (1/phi))
8	105.04759747124506	=	(34	*	sqrt 5) +	21	*	(2 - (1/phi))
9	169.97058314499202	=	(55	*	sqrt 5) +	34	*	(2 – (1/phi))

...
For n = 0, the constant is sqrt 5 (that is the Lucas per Fibonacci constant)
The multiplier of the first part of the formula is Fs, and of the second part is Fs-1 – this structure is similar to the Fibonacci per Fibonacci formula (.
Some numerical explanation: The difference between constants of n=2 and n=1 (between the 2nd and the 1st constant):
5.854101966249685 - 3.618033988749895 = 2.23606797749979 (= the 0th constant);
2 - (1/phi) = 2 - (1/1.618033988749895) = 1.38196601125, and
2.23606797749979 + 1.38196601125 = 3.618033988749895 (=the 1st constant)

Conclusions and closing remarks
To sum it up:

1. *we can generate a series of constants using the division that resulted in phi (s**ee: 1.** Fibonacci per Fibonacci constant)*
2. *there is a constant regarding the proportion of the Lucas and Fibonacci numbers and its value is square root of 5 (**see: 2.** Lucas per Fibonacci constant). This concept can be generalized for the Lucas per Fibonacci number divisions – and the result is a series of constants (**see: 3.** Generalization of Lucas per Fibonacci constant)*
3. *regarding these constants and generalizations, there are manifest mathematical patterns based on the phi behind these constants*

Obviously, these researches can be extended to other combinations of the Fibonacci sequence and Lucas series. For example, we generalized the Lucas per Fibonacci constant, but this is only one of the possible version:

L2/ F1; L3 /F2; L4/F3… (done)
L2/F3; L3/F4; L4/F5…
F2/L1; F3/L2; L4/f3…
F2/L3; F3/F4; F4/L5…
Etc.
We have not studied the features of the Padovan, Perrin, etc. lists from this point of view yet.
So this seems to be a promising research area.

SOURCES:
Lucas number
Weisstein, Eric W. "Lucas Number." From MathWorld--A Wolfram Web Resource.
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Fibonacci number
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http://mathworld.wolfram.com/FibonacciNumber.html

Fibonacci Numbers and The Golden Selection
http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/fib.html

The numerical results are based on programs written by me.

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