Modified and extended Ducci sequences and their inner patterns

Zoltán Galántai

1. Summary

A Ducci sequence can be interpreted as a subset of a broader category where not only the a < b < c < d order is allowed, but all of the possible order of the positive integers with different sizes. They are called "modified Ducci sequences". Similarly, the notion of the "extended Ducci sequences" can be introduced adding one or more elements to the original, four-component Ducci sequence in ascending order (e.g. a < b < c < d < e).

Conjectures about modified Ducci sequences:

There are only some possible repeating patterns for the modified Ducci sequences (e.g. for an "a,b,d,c" order). Besides the "0,0,0,0" version (this is the repeating pattern of the original Ducci sequences) only the next ones exist where "a" is a positive integer:

- a, 0, a, 0
- "a, 2a, a, 0
- 0, a, 0, a
- 2a, a, 0, a"

There is not any other possible pattern in the case of an extended, four-component Ducci sequence and probably this is true for any four positive integers.

Conjectures about extended Ducci sequences:

Next to the original Ducci sequence, there is one and one version where the repeating pattern consists only zeros: it is an extended version with 8 starting numbers.

Extended Ducci sequences with more than 8 starting numbers never end in repeating cycles although they have some more or less regular patterns with different numbers.

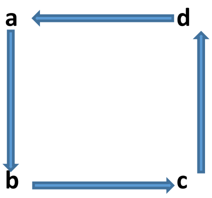
2. Modified Ducci sequences and their repeating patterns

According to the definition of Ducci sequences, writing down four integers in ascending order, and calculating the absolute value of them (a – b, b – c, c – d, d – a), the results will iterate to 0, 0, 0, 0, e.g.:

Differences of absolute values:

step	numbers	
0.	2, 5, 7, 13	(starting sequence)
1.	3, 2, 6, 11	
2.	3, 1, 3, 7	
3.	2, 2, 4, 4	
4.	0, 2, 0, 2	
5.	2, 2, 2, 2	
6.	0, 0, 0, 0	(repeating pattern)

A Ducci sequence can be described as numbers arranged in a square.



Four numbers (a, b, c, d) have 24 different arrangements, but as this illustration can show, the a, b, c, d arrangement is equivalent to three others: a b c d = b c d a = c d a b = d a b c

So we can categorize all the possible arrangements:

	1	2	3	4
1.	a b c d =	b c d a =	c d a b =	d a b c
2.	a b d c =	b d c a =	d c a b =	c a b d
3.	a c b d =	c b d a =	b d a c =	d a c b
4.	a c d b =	c d b a =	d b a c =	b a c d
5.	a d b c =	d b c a =	b c a d =	c a d b
6.	a d c b =	d c b a =	c b a d =	b a d c

Thus, “a b c d” is Ducci 1.1; “b c d a” is Ducci 1.2, etc. However, this does not mean that the repeating patterns are the same for the subcategories of the main categories (e.g. for 2.1 and 2.2; see below).

Results after 100,000 random number combinations (here and later: with randomly chosen numbers between 1 and 100; the combinations with two or more equal numbers were omitted):

Distributions of repeating parts:

[a]					
1.1 (a b c d)	A: 100000	B: 0	C: 0	D: 0	E: 0
1.2 (b c d a)	A: 100000	B: 0	C: 0	D: 0	E: 0
1.3 (c d a b)	A: 100000	B: 0	C: 0	D: 0	E: 0
1.4 (d a b c)	A: 100000	B: 0	C: 0	D: 0	E: 0
2.2 (b c d a)	A: 100000	B: 0	C: 0	D: 0	E: 0
2.3 (c d a b)	A: 100000	B: 0	C: 0	D: 0	E: 0
2.4 (d a b c)	A: 100000	B: 0	C: 0	D: 0	E: 0
6.1 (a d c b)	A: 100000	B: 0	C: 0	D: 0	E: 0
6.2 (d c b a)	A: 100000	B: 0	C: 0	D: 0	E: 0
6.3 (c b a d)	A: 100000	B: 0	C: 0	D: 0	E: 0
6.4 (b a d c)	A: 100000	B: 0	C: 0	D: 0	E: 0
[b]					
2.1 (a b d c)	A: 1618	B: 73028	C: 25354	D: 0	E: 0
3.2 (c b d a)	A: 1446	B: 62652	C: 35902	D: 0	E: 0
5.1 (a d b c)	A: 1707	B: 80316	C: 17977	D: 0	E: 0
[c]					
3.1 (a c b d)	A: 1801	B: 0	C: 0	D: 80254	E: 17945
4.1 (a c d b)	A: 1768	B: 0	C: 0	D: 78789	E: 19443
5.4 (c a d b)	A: 1445	B: 0	C: 0	D: 62829	E: 35726
[d]					
3.3 (b d a c)	A: 3815	B: 0	C: 0	D: 0	E: 96185
4.3 (d b a c)	A: 6237	B: 0	C: 0	D: 0	E: 93763
5.2 (d b c a)	A: 8925	B: 0	C: 0	D: 0	E: 91075
[e]					
3.4 (d a c b)	A: 8763	B: 0	C: 91237	D: 0	E: 0
4.2 (c d b a)	A: 2024	B: 0	C: 97976	D: 0	E: 0
4.4 (b a c d)	A: 1981	B: 0	C: 98019	D: 0	E: 0
5.3 (b c a d)	A: 3810	B: 0	C: 96190	D: 0	E: 0

Explanation of symbols:	e.g.:
A = 0, 0, 0, 0	0,0,0,0
B = a, 2a ,a, 0	1,2,1,0
C = a, 0, a, 0	3,0,3,0
D = 2a, a, 0, a	4 2 0 2
E = 0, a, 0, a	0 3 0 3

Notice that all the Ducci-1 and Ducci-6 versions end only in the “0,0,0,0” pattern – similarly to the Ducci-2.2; Ducci-2.3 and Ducci-2.4 **[a]**. This is almost half (11) of the 24 Ducci-versions. The second most frequent distribution regarding the endings is **[e]**. This represents only the 1/6th of the appearing variations (Ducci-3.4; Ducci-4.2; Ducci-4.4 and Ducci-5.3); and the rest is divided proportionally between three other variations (**[b]**, **[c]**, **[d]**). However, there is no a Ducci version where the endings proportionally distributed between A and B; or A, and B, and C, and D, and E; where every ending is only B, or C…, etc. In short, instead of 5!, we have only five distributions.

3. Extended Ducci sequences

If we modify the original Ducci 1 sequence adding new element(or new elements to it ("a,b,c,d,e" or "a,b,c,d,e,f,g,h,i,j,k"), we get new patterns. In the case of the extended Ducci sequence with five numbers, the repeating pattern is not only one but 15 lines long.

Besides the repeating sequence built form zeros and ones, there are variations with twos and zeros; threes and zeros, and so on. These lines are the components the repeating patterns, but they do not necessarily begin with the "1 1 1 1 0 0" set. In some cases, the 1s are replaced with 2s, 3s, 4s, 5s, 6s, and so on.

Extended Ducci with 5 components (examples):

step	numbers		numbers	
0.	5 15 21 38 45		23 25 81 83 91	
1.	10 6 17 7 40		2 56 2 8 68	
2.	4 11 10 33 30		54 54 6 60 66	
3.	7 1 23 3 26		0 48 54 6 12	
4.	6 22 20 23 19		48 6 48 6 12	
5.	16 2 3 4 13		42 42 42 6 36	
6.	14 1 1 9 3		0 0 36 30 6	
7.	13 0 8 6 11		0 36 6 24 6	
8.	13 8 2 5 2		36 30 18 18 6	
9.	5 6 3 3 11		6 12 0 12 30	
10.	1 3 0 8 6		6 12 12 18 24	
11.	2 3 8 2 5		6 0 6 6 18	
12.	1 5 6 3 3		6 6 0 12 12	
13.	4 1 3 0 2		0 6 12 0 6	
14.	3 2 3 2 2		6 6 12 6 6	
15.	**1 1 1 0 1**	(repeating pattern)	**0 6 6 0 0**	(repeating pattern)
16.	**0 0 1 1 0**		**6 0 6 0 0**	
17.	**0 1 0 1 0**		**6 6 6 0 6**	
18.	**1 1 1 1 0**		**0 0 6 6 0**	
19.	**0 0 0 1 1**		**0 6 0 6 0**	
20.	**0 0 1 0 1**		**6 6 6 6 0**	
21.	**0 1 1 1 1**		**0 0 0 6 6**	
22.	**1 0 0 0 1**		**0 0 6 0 6**	
23.	**1 0 0 1 0**		**0 6 6 6 6**	
24.	**1 0 1 1 1**		**6 0 0 0 6**	
25.	**1 1 0 0 0**		**6 0 0 6 0**	
26.	**0 1 0 0 1**		**6 0 6 6 6**	
27.	**1 1 0 1 1**		**6 6 0 0 0**	
28.	**0 1 1 0 0**		**0 6 0 0 6**	
29.	**1 0 1 0 0**		**6 6 0 6 6**	

Examining 1 million random example:

components	how many
0 and 1	881859
0 and 2	50449
0 and 3	59103
0 and 4	2523
0 and 5	875
0 and 6	3134
0 and 7	197
0 and 8	94
0 and 9	1394
0 and 10	30
0 and other	342

Extended Ducci with 6 and 7 components:

Ducci 6		Ducci 7	
step	numbers	step	numbers
1.	4 33 48 57 62 68	0.	3 52 55 55 80 97 99
2.	29 15 9 5 6 64	1.	49 3 0 25 17 2 96
3.	14 6 4 1 58 35	2.	46 3 25 8 15 94 47
4.	8 2 3 57 23 21	3.	43 22 17 7 79 47 1
5.	6 1 54 34 2 13	4.	21 5 10 72 32 46 42
6.	5 53 20 32 11 7	5.	16 5 62 40 14 4 21
7.	48 33 12 21 4 2	6.	11 57 22 26 10 17 5
8.	15 21 9 17 2 46	7.	46 35 4 16 7 12 6
9.	6 12 8 15 44 31	8.	11 31 12 9 5 6 40
10.	6 4 7 29 13 25	9.	20 19 3 4 1 34 29
11.	2 3 22 16 12 19	10.	1 16 1 3 33 5 9
12.	1 19 6 4 7 17	11.	15 15 2 30 28 4 8
13.	18 13 2 3 10 16	12.	0 13 28 2 24 4 7
14.	5 11 1 7 6 2	13.	13 15 26 22 20 3 7
15.	6 10 6 1 4 3	14.	2 11 4 2 17 4 6
16.	4 4 5 3 1 3	15.	9 7 2 15 13 2 4
17.	0 1 2 2 2 1	16.	2 5 13 2 11 2 5
		17.	3 8 11 9 9 3 3
18.	**1 1 0 0 1 1** (repeating pattern)	18.	5 3 2 0 6 0 0
19.	**0 1 0 1 0 0**	19.	2 1 2 6 6 0 5
20.	**1 1 1 1 0 0**	20.	1 1 4 0 6 5 3
21.	**0 0 0 1 0 1**	21.	0 3 4 6 1 2 2
22.	**0 0 1 1 1 1**	22.	3 1 2 5 1 0 2
23.	**0 1 0 0 0 1**	23.	2 1 3 4 1 2 1
		24.	1 2 1 3 1 1 1
		25.	1 1 2 2 0 0 0
		26.	0 1 0 2 0 0 1
		27.	1 1 2 2 0 1 1
		28.	0 1 0 2 1 0 0
		29.	1 1 2 1 1 0 0
		30.	**0 1 1 0 1 0 1** (repeating pattern)
		31.	**1 0 1 1 1 1 1**
		32.	**1 1 0 0 0 0 0**
		33.	**0 1 0 0 0 0 1**
		34.	**1 1 0 0 0 1 1**
		35.	**0 1 0 0 1 0 0**
		36.	**1 1 0 1 1 0 0**

Ducci 8:

1. 7 48 54 63 68 73 75 93
2. 41 6 9 5 5 2 18 86
3. 35 3 4 0 3 16 68 45
4. 32 1 4 3 13 52 23 10
5. 31 3 1 10 39 29 13 22
6. 28 2 9 29 10 16 9 9
7. 26 7 20 19 6 7 0 19
8. 19 13 1 13 1 7 19 7
9. 6 12 12 12 6 12 12 12
10. 6 0 0 6 6 0 0 6
11. 6 0 6 0 6 0 6 0
12. 6 6 6 6 6 6 6 6

13. 0 0 0 0 0 0 0 0

Ducci 9:

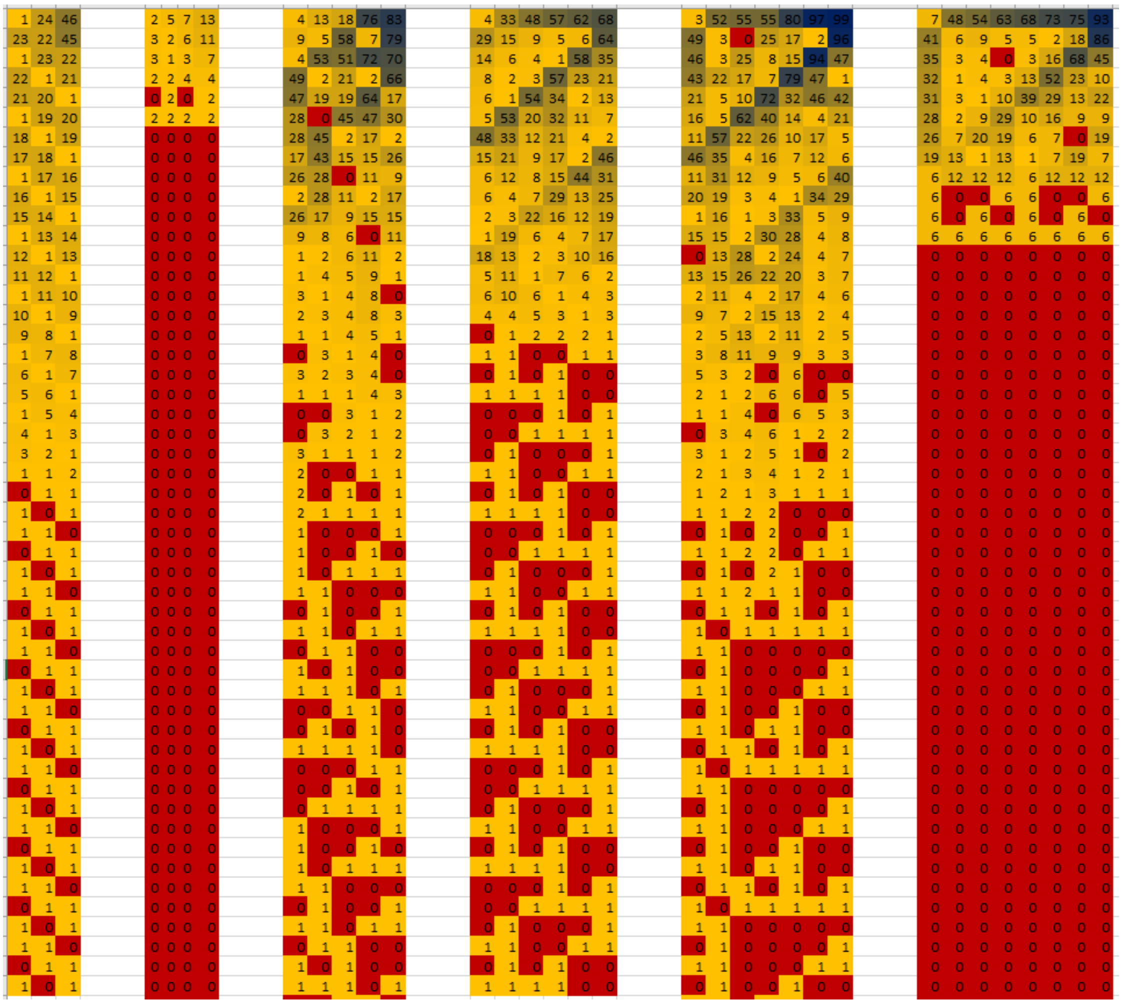
While the repeating pattern of a Ducci with 8 numbers is similar to a Ducci 4's since this contains only zeros, the Ducci 9 represents a fundamental change. Examining to 50 million steps, we did not find any repeating pattern. If our conjecture is true, then there is no Ducci sequence with more than 8 starting numbers with identical repeating lines.

4. Appendix

1. **Patterns of extended Ducci version: some examples**

Number of the components

3: **4:** **5:** **6:** **7:** **8:**



9:

3	8	13	19	30	30	52	54	94
5	5	6	11	0	22	2	53	2
0	1	5	11	22	20	51	51	3
1	4	6	11	2	31	0	48	3
3	2	5	9	29	31	48	44	3
1	3	4	20	2	17	4	39	2
2	1	16	18	15	13	35	33	5
1	15	2	3	2	22	2	26	5
14	13	1	1	20	20	24	18	7
1	12	0	19	0	4	6	9	5
11	12	19	19	4	2	3	1	9
1	7	0	15	2	1	2	10	0
6	7	15	13	1	1	8	2	11
1	8	2	12	0	7	6	11	7
7	6	10	12	7	1	5	3	13
1	4	2	5	6	4	2	12	8
3	2	3	1	2	2	10	4	15
1	1	2	1	0	8	6	13	14
0	1	1	1	8	2	7	5	17
1	0	0	7	6	5	2	14	19
1	0	7	1	1	3	12	6	19
1	7	6	0	2	9	6	15	20
6	1	6	2	7	3	9	7	21
5	5	4	5	4	6	2	16	17
0	1	1	1	2	4	14	8	19
1	0	0	1	2	10	6	17	25
1	0	1	1	8	4	11	9	25
1	1	0	7	4	7	2	18	26
0	1	7	3	3	5	16	10	27
1	6	4	0	2	11	6	19	29
5	2	4	2	9	5	13	11	29
3	2	2	7	4	8	2	20	26
1	0	5	3	4	6	18	12	29
1	5	2	1	2	12	6	21	32
4	3	1	1	10	6	15	13	33
1	2	0	9	4	9	2	22	31
1	2	9	5	5	7	20	14	35
1	7	4	0	2	13	6	23	36
6	3	4	2	11	7	17	15	37
3	1	2	9	4	10	2	24	33
2	1	7	5	6	8	22	16	37
1	6	2	1	2	14	6	25	39
5	4	1	1	12	8	19	17	41
1	3	0	11	4	11	2	26	38
2	3	11	7	7	9	24	18	43
1	8	4	0	2	15	6	27	43
7	4	4	2	13	9	21	19	45
3	0	2	11	4	12	2	28	40
3	2	9	7	8	10	26	20	45
1	7	2	1	2	16	6	29	46
6	5	1	1	14	10	23	21	49

10:

17	31	35	37	61	68	71	97	99	99
14	4	2	24	7	3	26	96	98	82
10	2	22	17	4	23	70	94	80	68
8	20	5	13	19	47	24	91	65	58
12	15	8	6	28	23	67	87	54	50
3	7	2	22	5	44	20	82	45	38
4	5	20	17	39	24	62	76	32	35
1	15	3	22	15	38	14	69	28	31
14	12	19	7	23	24	55	61	23	30
2	7	12	16	1	31	6	52	21	16
5	5	4	15	30	25	46	42	6	14
0	1	11	15	5	21	4	31	3	9
1	10	4	10	16	17	27	19	3	9
9	6	6	6	1	10	8	6	4	8
3	0	0	5	9	2	2	8	6	1
3	0	5	4	7	0	6	7	14	2
3	5	1	3	7	6	1	9	14	1
2	4	2	4	1	5	8	8	16	2
2	2	2	3	4	3	0	10	16	0
0	0	1	1	1	3	10	9	19	2
0	1	0	0	2	7	1	11	18	2
1	1	0	2	5	6	10	10	19	2
0	1	2	3	1	4	0	12	20	1
1	1	1	2	3	4	12	11	22	1
0	0	1	1	1	8	1	13	23	0
0	1	0	0	7	7	12	12	25	0
1	1	0	7	0	5	0	14	26	0
0	1	7	7	5	5	14	13	27	1
1	6	0	2	0	9	1	15	27	1
5	6	2	2	9	8	14	14	28	0
1	4	0	7	1	6	0	16	30	5
3	4	7	6	5	6	16	15	26	4
1	3	1	1	1	10	1	17	28	1
2	2	0	0	9	9	16	16	32	0
0	2	0	9	0	7	0	18	34	2
2	2	9	9	7	7	18	17	33	2
0	7	0	2	0	11	1	19	34	0
7	7	2	2	11	10	18	18	37	0
0	5	0	9	1	8	0	20	38	7
5	5	9	8	7	8	20	19	32	7
0	4	1	1	1	12	1	21	33	2
4	3	0	0	11	11	20	20	39	2
1	3	0	11	0	9	0	22	40	2
2	3	11	11	9	9	22	21	41	1
1	8	0	2	0	13	1	23	43	1
7	8	2	2	13	12	22	22	44	0
1	6	0	11	1	10	0	24	46	7
5	6	11	10	9	10	24	23	40	6
1	5	1	1	1	14	1	25	42	1
4	4	0	0	13	13	24	24	48	0
0	4	0	13	0	11	0	26	50	4

11:

1	35	37	41	51	61	64	73	76	82	82
34	2	4	10	10	3	9	72	81	0	81
32	2	6	0	7	6	63	70	2	81	47
30	4	6	7	1	57	7	67	78	34	15
26	2	1	6	56	50	60	63	30	19	15
24	1	5	50	6	10	3	58	14	4	11
23	4	45	44	4	7	55	52	2	7	13
19	41	1	40	3	48	3	45	0	6	10
22	40	39	37	45	45	42	37	2	4	9
18	1	2	8	0	3	5	28	5	5	13
17	1	6	8	3	2	23	18	5	8	5
16	5	2	5	1	21	5	7	3	3	12
11	3	3	4	20	16	2	5	9	9	4
8	0	1	16	4	14	3	8	4	5	7
8	1	15	12	10	11	5	6	9	2	1
7	14	3	2	1	6	1	9	13	1	7
7	11	1	1	5	5	8	7	15	6	0
4	10	0	4	0	3	1	10	11	6	7
6	10	4	4	3	2	9	8	12	1	3
4	6	0	1	1	7	1	11	18	2	3
2	6	1	0	6	6	10	9	18	1	1
4	5	1	6	0	4	1	12	20	0	1
1	4	5	6	4	3	11	10	22	1	3
3	1	1	2	1	8	1	13	22	2	2
2	0	1	1	7	7	12	11	22	0	1
2	1	0	6	0	5	1	14	25	1	1
1	1	6	6	5	4	13	12	25	0	1
0	5	0	1	1	9	1	15	27	1	0
5	5	1	0	8	8	14	13	27	1	0
0	4	1	8	0	6	1	16	28	1	5
4	3	7	8	6	5	15	14	29	4	5
1	4	1	2	1	10	1	17	27	1	1
3	3	1	1	9	9	16	15	31	0	0
0	2	0	8	0	7	1	18	33	0	3
2	2	8	8	7	6	17	16	34	3	3
0	6	0	1	1	11	1	19	32	0	1
6	6	1	0	10	10	18	17	36	1	1
0	5	1	10	0	8	1	20	36	0	5
5	4	9	10	8	7	19	18	38	5	5
1	5	1	2	1	12	1	21	34	0	0
4	4	1	1	11	11	20	19	40	0	1
0	3	0	10	0	9	1	22	41	1	3
3	3	10	10	9	8	21	20	41	2	3
0	7	0	1	1	13	1	23	41	1	0
7	7	1	0	12	12	22	21	43	1	0
0	6	1	12	0	10	1	24	44	1	7
6	5	11	12	10	9	23	22	45	6	7
1	6	1	2	1	14	1	25	41	1	1
5	5	1	1	13	13	24	23	47	0	0
0	4	0	12	0	11	1	26	49	0	5
4	4	12	12	11	10	25	24	50	5	5

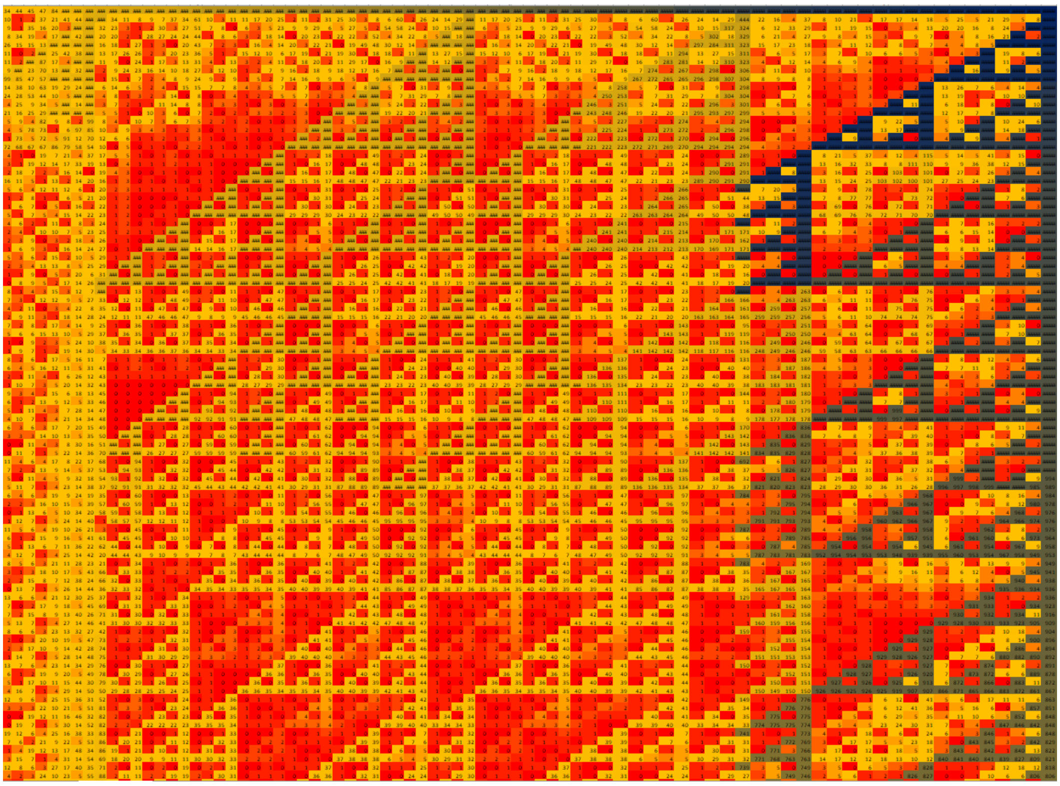
12:

23	24	26	26	39	51	52	54	72	80	80	83
1	2	0	13	12	1	2	53	79	0	3	60
1	2	13	1	11	1	51	51	2	3	57	59
1	11	12	10	10	50	0	48	0	54	2	58
10	1	2	0	40	50	48	44	50	52	56	57
9	1	2	40	10	2	4	39	47	4	1	47
8	1	38	30	8	2	35	33	2	3	46	38
7	37	8	22	6	33	2	26	4	43	8	30
30	29	14	16	27	31	24	18	35	35	22	23
1	15	2	11	4	7	6	9	26	13	1	7
14	13	9	7	3	1	3	1	3	12	6	6
1	4	2	4	2	2	2	10	1	6	0	8
3	2	2	2	0	0	8	2	6	6	8	7
1	0	0	2	0	8	6	11	7	2	1	4
1	0	2	2	8	2	5	3	12	1	3	3
1	2	0	6	6	3	2	12	14	2	0	2
1	2	6	0	3	1	10	4	14	2	2	1
1	4	6	3	2	9	6	13	15	0	1	0
3	2	3	1	7	3	7	5	18	1	1	1
1	1	2	6	4	4	2	14	18	0	0	2
0	1	4	2	0	2	12	6	20	0	2	1
1	3	2	2	2	10	6	15	21	2	1	1
2	1	0	0	8	4	9	7	20	1	0	0
1	1	0	8	4	5	2	16	22	1	0	2
0	1	8	4	1	3	14	8	23	1	2	1
1	7	4	3	2	11	6	17	24	1	1	1
6	3	1	1	9	5	11	9	25	0	0	0
3	2	0	8	4	6	2	18	27	0	0	6
1	2	8	4	2	4	16	10	28	0	6	3
1	6	4	2	2	12	6	19	29	6	3	2
5	2	2	0	10	6	13	11	24	3	1	1
3	0	2	10	4	7	2	20	28	2	0	4
3	2	8	6	3	5	18	12	30	2	4	1
1	6	2	3	2	13	6	21	31	2	3	2
5	4	1	1	11	7	15	13	32	1	1	1
1	3	0	10	4	8	2	22	34	0	0	4
2	3	10	6	4	6	20	14	36	0	4	3
1	7	4	2	2	14	6	23	37	4	1	1
6	3	2	0	12	8	17	15	34	3	0	0
3	1	2	12	4	9	2	24	36	3	0	6
2	1	10	8	5	7	22	16	37	3	6	3
1	9	2	3	2	15	6	25	38	3	3	1
8	7	1	1	13	9	19	17	39	0	2	0
1	6	0	12	4	10	2	26	43	2	2	8
5	6	12	8	6	8	24	18	42	0	6	7
1	6	4	2	2	16	6	27	45	6	1	2
5	2	2	0	14	10	21	19	40	5	1	1
3	0	2	14	4	11	2	28	42	4	0	4
3	2	12	10	7	9	26	20	44	4	4	1
1	10	2	3	2	17	6	29	45	0	3	2
9	8	1	1	15	11	23	21	50	3	1	1

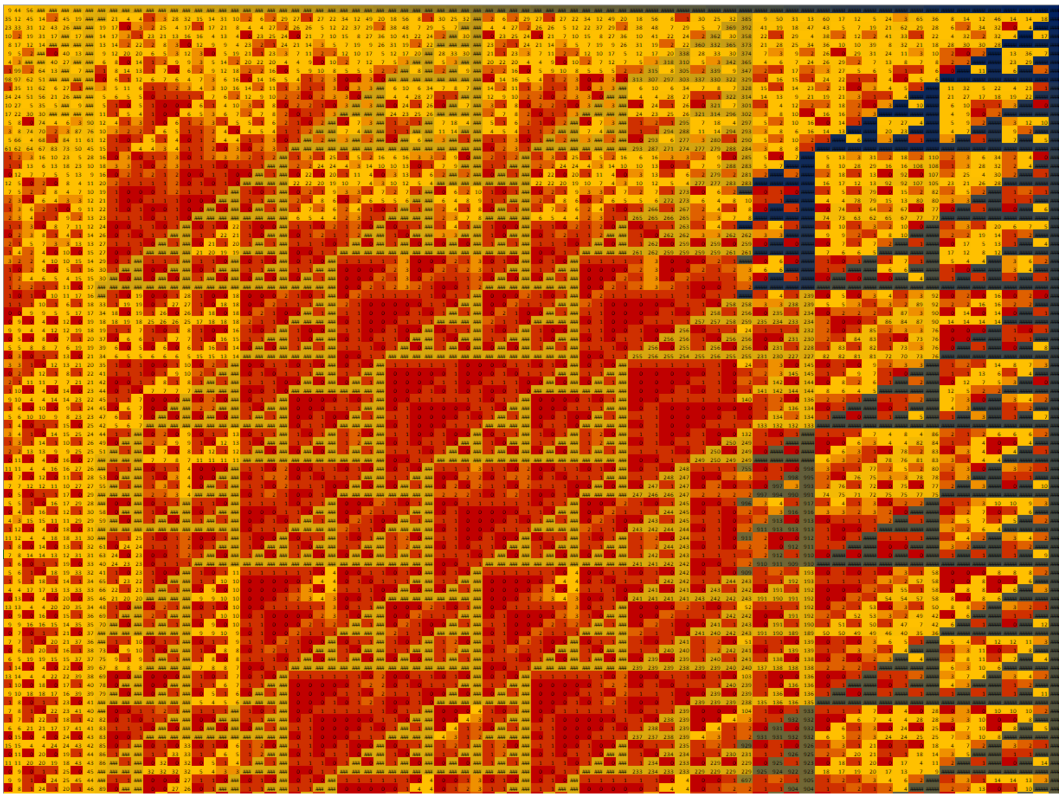
40:

4	17	24	31	33	43	66	109	118	131	182	185	197	229	234	261	291	354	356	366	386	391	396	405	425	426	456	466	488	532	552	590	598	611	627	654	663	672	684	692
13	7	7	2	10	23	43	108	130	51	3	12	32	5	27	30	63	2	10	20	5	5	9	20	1	30	10	22	44	20	38	8	13	16	27	9	9	12	8	688
6	0	5	8	13	20	65	106	49	48	9	20	27	22	3	33	61	8	10	15	0	4	11	19	29	20	12	22	24	18	30	5	3	11	18	0	3	4	680	675
6	5	3	5	7	45	41	103	45	39	11	7	5	19	30	28	53	2	5	15	4	7	8	10	9	8	10	2	6	12	25	2	8	7	18	3	1	676	5	669
1	2	2	2	38	4	62	99	35	28	4	2	14	11	2	25	51	3	10	11	3	1	2	1	1	2	8	4	6	13	23	6	1	11	15	2	675	671	664	663
1	0	0	36	34	58	37	94	23	24	2	12	3	9	23	26	48	7	1	8	2	1	1	0	1	6	4	2	7	10	17	5	10	4	13	673	4	7	1	662
1	0	36	2	24	21	57	88	18	22	10	9	6	14	3	22	41	6	7	6	1	0	1	1	5	2	2	5	3	7	12	5	6	9	660	669	3	6	661	661
1	36	34	22	3	36	31	81	15	12	1	3	8	11	19	19	35	1	1	5	1	1	0	4	3	0	3	2	4	5	7	1	3	651	9	666	3	655	0	660
35	2	12	19	33	5	50	73	4	11	2	5	3	8	0	16	34	0	4	4	0	1	4	1	3	3	1	2	1	2	6	2	648	642	657	663	652	655	660	659
33	10	7	14	28	45	23	64	2	9	3	2	5	8	16	18	34	4	0	4	1	3	3	2	0	2	1	1	1	4	4	646	6	15	6	11	3	5	1	624
23	3	7	14	17	22	41	54	1	6	1	3	3	8	2	16	30	4	4	3	2	0	1	2	2	1	0	0	3	0	642	640	9	9	5	8	2	4	623	591
20	4	7	3	5	19	13	43	5	5	2	0	5	6	14	14	26	0	1	1	2	1	1	0	1	1	0	3	3	642	2	631	0	4	3	6	2	619	32	568
16	3	4	2	14	6	30	31	7	3	2	5	1	8	0	12	26	1	0	1	1	0	1	1	0	1	3	0	639	640	629	631	4	1	3	4	617	587	536	548
13	1	2	12	8	24	1	18	10	1	3	4	7	8	12	14	25	1	1	0	1	1	0	1	1	2	3	639	1	11	2	627	3	2	1	613	30	51	12	532
12	1	10	4	16	23	17	4	13	2	1	3	1	4	2	11	24	0	1	1	0	1	1	0	1	1	636	638	10	9	625	624	1	1	612	583	21	39	520	519
11	9	6	12	7	6	13	11	13	1	2	2	3	2	9	13	24	1	0	1	1	0	1	1	0	635	2	628	1	616	1	623	0	611	29	562	18	481	1	507
2	3	6	5	1	7	2	5	15	1	0	1	1	7	4	11	23	1	1	0	1	1	0	1	635	633	626	627	615	615	622	623	611	582	533	544	463	480	506	496
1	3	1	4	6	5	3	12	16	1	1	0	6	3	7	12	22	0	1	1	0	1	1	634	2	7	1	12	0	7	1	12	29	49	11	81	17	26	10	494
2	2	3	2	1	2	9	6	17	0	1	6	3	4	5	10	22	1	0	1	1	0	633	632	5	6	11	12	7	6	11	17	20	38	70	64	9	16	484	493
0	1	1	1	1	7	3	13	19	1	5	3	1	1	5	12	21	1	1	0	1	633	1	627	1	5	1	5	1	5	6	3	18	32	6	55	7	468	9	491
1	0	0	0	6	4	10	7	19	4	2	2	0	4	7	9	20	0	1	1	632	632	626	626	4	4	4	4	4	1	3	15	14	26	49	48	461	459	482	491
1	0	0	6	2	6	3	14	17	2	0	2	4	3	2	11	20	1	0	631	0	6	0	622	0	0	0	0	3	2	12	1	12	23	1	413	2	23	9	490
1	0	6	4	4	3	11	8	20	2	2	2	1	1	9	9	19	1	631	631	6	6	622	622	0	0	0	3	1	10	11	11	11	22	412	411	21	14	481	489
1	6	2	0	1	8	3	15	21	0	0	1	0	8	0	10	18	630	0	625	0	616	0	622	0	0	3	2	9	1	0	0	11	390	1	390	7	467	8	488
5	4	2	1	7	5	12	9	24	0	1	1	8	8	10	8	612	630	625	625	616	616	622	622	0	3	1	7	8	1	0	11	379	389	389	383	460	459	480	487
1	2	1	6	2	7	3	16	25	1	0	7	0	2	2	604	18	5	0	9	0	6	0	622	3	2	6	1	7	1	11	368	10	0	6	77	1	21	7	482
1	1	5	4	5	4	13	10	25	1	7	7	2	0	602	586	13	5	9	9	6	6	622	619	1	4	5	6	6	10	357	358	10	6	71	76	20	14	475	481
0	4	1	1	1	9	3	17	26	6	0	5	2	602	16	573	8	4	0	3	0	616	3	618	3	1	1	0	4	347	1	348	4	65	5	56	6	461	6	480
4	3	0	0	8	6	14	11	22	6	5	3	600	586	557	565	4	4	3	3	616	613	615	615	2	0	1	4	343	346	347	344	61	60	51	50	455	455	474	480
1	3	0	8	2	8	3	18	23	1	2	597	14	29	8	561	0	1	0	613	3	2	0	613	2	1	3	339	3	1	3	283	1	9	1	405	0	19	6	476
2	3	8	6	6	5	15	12	29	1	595	583	15	21	553	561	1	1	613	610	1	2	613	611	1	2	336	336	2	2	280	282	8	8	404	405	19	13	470	475
1	5	2	0	1	10	3	19	30	594	12	568	6	532	8	560	0	612	3	609	1	611	2	610	1	334	0	334	0	278	2	274	0	396	1	386	6	457	5	473
4	3	2	1	9	7	16	13	562	582	556	562	526	524	552	560	612	609	606	608	610	609	608	609	333	334	334	334	278	276	272	274	396	395	385	380	451	452	468	472
1	1	1	8	2	9	3	20	549	26	6	36	2	28	8	52	3	3	2	2	1	1	1	276	1	0	0	56	2	4	2	122	1	10	5	71	1	16	4	468
0	0	7	6	7	6	17	14	8	20	30	34	26	20	44	49	0	1	0	1	0	0	275	275	1	0	56	54	2	2	120	121	9	5	66	70	15	12	464	467
0	7	1	1	1	11	3	21	15	10	4	8	6	24	5	49	1	1	1	1	0	275	0	274	1	56	2	52	0	118	1	112	4	61	4	55	3	452	3	467
7	6	0	0	10	8	18	15	26	6	4	2	18	19	44	48	0	0	0	1	275	275	274	273	55	54	50	52	118	117	111	108	57	57	51	52	449	449	464	467
1	6	0	10	2	10	3	22	31	2	2	16	1	25	4	48	0	0	1	274	0	1	1	218	1	4	2	66	1	6	3	51	0	6	1	397	0	15	3	460
5	6	10	8	8	7	19	16	36	0	14	15	24	21	44	48	0	1	273	274	1	0	217	217	3	2	64	65	5	3	48	51	6	5	396	397	15	12	457	459
1	4	2	0	1	12	3	23	39	14	1	9	3	23	4	48	1	272	1	273	1	217	0	214	1	62	1	60	2	45	3	45	1	391	1	382	3	445	2	454
3	2	2	1	11	9	20	17	26	13	8	6	20	19	44	47	271	271	272	272	216	217	214	213	61	61	59	58	43	42	42	44	390	390	381	379	442	443	452	453
1	0	1	10	2	11	3	24	28	5	2	14	1	25	3	224	0	1	0	56	1	3	1	152	0	2	1	15	1	0	2	346	0	9	2	63	1	9	1	450
1	1	9	8	9	8	21	18	37	3	12	13	24	22	221	224	1	1	56	55	2	2	151	152	2	1	14	14	1	2	344	346	9	7	61	62	8	8	449	449
0	8	1	1	1	13	3	25	40	9	1	11	2	199	3	223	0	55	1	53	0	149	1	150	1	13	0	13	1	342	2	337	2	54	1	54	0	441	0	448
8	7	0	0	12	10	22	19	35	8	10	9	197	196	220	223	55	54	52	53	149	148	149	149	12	13	13	12	341	340	335	335	52	53	53	54	441	441	448	448

80:



80:



2. Ducci versions: number of the steps to the repeating pattern

Regarding the numbers of the steps needed to reach the repeating pattern, the modified Ducci sequences can be divided into subcategories.
After about 20,000 repetition and with 10^{16} as the largest possible, randomly chosen number, the numbers of the steps to the repeating patterns where the starting four numbers is the 0^{th} step (the results rounded to one decimal):

2.2(b d c a)	4.0	4.0	[a]
2.4 (c a b d)	4.0	4.0	[a]
4.2 (c d b a)	4.0	4.0	[e]
4.4 (b a c d)	4.0	4.0	[e]
3.2 (c b d a)	5.525726286314316	5.5	[b]
3.3 (b d a c)	5.519925996299815	5.5	[d]
5.3 (b c a d)	5.514025701285064	5.5	[e]
5.4 (c a d b)	5.525576278813941	5.5	[c]
5.1 (a d b c)	5.738736936846842	5.7	[b]
5.2 (d b c a)	5.748087404370218	5.7	[d]
2.1 (a b d c)	6.279413970698535	6.3	[b]
2.3 (d c a b)	6.280514025701285	6.3	[a]
4.1 (a c d b)	6.2689134456722835	6.3	[c]
4.3 (d b a c)	6.334516725836292	6.3	[d]
1.1 (a b c d)	6.87874393719686	6.9	[a]
1.2 (b c d a)	6.873843692184609	6.9	[a]
1.3 (c d a b)	6.879093954697735	6.9	[a]
1.4 (d a b c)	6.874943747187359	6.9	[a]
6.1 (a d c b)	6.875393769688484	6.9	[a]
6.2 (d c b a)	6.874843742187109	6.9	[a]
6.3 (c b a d)	6.881694084704235	6.9	[a]
6.4 (b a d c)	6.869993499674984	6.9	[a]
3.1 (a c b d)	5.740537026851342	7.4	[c]
3.4 (d a c b)	5.736536826841342	7.4	[e]

As it can be seen, the connection between the lengths and the categories ([a], [b]…) is usually weak or does not exist at all. The only manifest exceptions are the Ducci-1 and Ducci-6 versions: all of them belong to the 6.9 length and no other modified Ducci belongs to it. However, even the [a] category is inhomogeneous since this contains elements with 4.0 and 6.3 length as well:

2.2(b d c a)	4.0	4.0	[a]
2.4 (c a b d)	4.0	4.0	[a]
2.3 (d c a b)	6.280514025701285	6.3	[a]

The cases of the other categories are similar. See, for example, the [b] category:

3.2 (c b d a)	5.525726286314316	5.5	[b]
5.1 (a d b c)	5.738736936846842	5.7	[b]
2.1 (a b d c)	6.279413970698535	6.3	[b]

Sources:

- Chamberland, Marc: Single Digits. Princeton Univ. Press, 2015
- Programs were written by me.