On the Fine Structure of Shifted Padovan, Perrin, and Fibonacci Sequences and Lucas Series

Abstract

The aim of this short paper to introduce the notion of the shifted sequences; and to describe the mathematical rules determine the constants of the shifted sequences of the Padovan, Perrin and Fibonacci sequences and Lucas series. These new constants (similar to to plastic number) are listed.

1. Introduction and some notations

Let P1, P2, P3, Pn… (P1 < P2 <P3) the first, second, third, nth item of either the Perrin or the Padovan sequences. Obviously, p1/p1; P2/P2, P3/P3… Pn/Pn = 1.
Dividing P2/P1, P3/P2, P4/P3… pn+1/pn, the results converge to a constant named plastic number (1.3247179572465302. Notice that all of these constants are approximations).
If we divide P3/P1; P4/P2; P5/P3... Pn+2/Pn, then the results converge to another constant (1.7548776662467724); and, similarly, we can divide P4/P1; P5/P2; P6/P3... etc. A neologism can be introduced: we can call these these sequences to shifted sequences, where the degree of the shifting is 0 in the first case (P1/P1; P2/P2; P3/P3).
The difference between the serial number of the numerator and the serial number of the denominator is the degree of the shifted sequence; e.g. the degree of the Perrin sequence (where the constant is the plastic number) is 1 while the degree of the P1/P2, P2/P3, P3/P4 sequence is −1. So there are two different types of sequences: P+/P and P−/P : in short form, P+ and P−.

2. Shifted Peerin (and Padovan) sequences

2.1 P+ sequences

We do not distinguish between the Perrin and Padovan sequences since their ratios converge to the same numbers.
Changing the degree, the result is a constant in every case and the values of the constants are governed rules based on and exclusively on the Perrin sequence.

degree			a	b		constant	
0						1.0	
1	x^3	−	***0*** x^2	***−1***x	−1	1.3247179572465302	(plastic number)
2	x^3	−	***2*** x^2	***+1***x	−1	1.7548776662467724	
3	x^3	−	***3*** x^2	***+2***x	−1	2.32471795724653	
4	x^3	−	***2*** x^2	***−3***x	−1	3.0795956234933026	
5	x^3	−	***5*** x^2	***+4***x	−1	4.079595623493303	
6	x^3	−	***5*** x^2	***−2***x	−1	5.404313580739832	
7	x^3	−	***7*** x^2	***−1***x	−1	7.159191246986605	
8	x^3	−	***10*** x^2	***+5***x	−1	9.483909204233136	
9	x^3	−	***12*** x^2	***−7***x	−1	12.563504827726439	
10	x^3	−	***17*** x^2	***+6***x	−1	16.64310045121974	

…

From degree 1, the formula describing the values of the constants:
$x^3 - a * x^2 + b * x - 1$ = constant where
a = 0, +2,+3,+2,+5,+5,+7,+10,+12,+17...
These are the elements of the Perrin sequence from P2.
B = −1,+1,+2,-3,+4,−2,−1,+5,−7,+ 6...
The elements of the Perrin sequence for negative numbers from P2 are:
−P = +1,−1,−2, +3,−4, 2, +1,−5,+7,−6 (OEIS A078712)
Thus the values of the **b** items equal to **−P *-1**:

b	−1	+1	+2	-3	+4	-2	−1	+5	-7	+6
−P	+1	−1	-2	+3	-4	+2	+1	-5	+7	-6

It is remarkable that only the Perrin sequence has a role in the formation of these constants, although these constants are the same for both the shifted Padovan and shifted Perrin sequences. Poetically speaking, the Padovan is only the shadow of the Perrin sequence. The same is true for the sequences of P–.

2.2 P− SEQUENCES

degree			a		b	constant
0						1.0
−1	x^3	+	***1***x^2	***+0***x	−1	0.7548776662466136
−2	x^3	−	***1***x^2	***+2***x	−1	0.5698402909981943
−3	x^3	−	***2***x^2	***+3***x	−1	0.430159709001921
−4	x^3	+	***3***x^2	***+2***x	−1	0.3247179572447772
−5	x^3	−	***4***x^2	***+5***x	−1	0.245122333753325
−6	x^3	+	***2***x^2	***+5***x	−1	0.18503737524864486
−7	x^3	+	***1***x^2	***+7***x	−1	0.13968058199611808
−8	x^3	−	***5***x^2	***+10***x	−1	0.10544175175720574
−9	x^3	+	***7***x^2	***+12***x	−1	0.07959562349144353
−10	x^3	−	***6***x^2	***+17***x	−1	0.060084958504671514

...

As a general rule about these shifted sequences (including the Fibonacci sequences and the Lucas series, see below), P– sequence = 1/P+ sequence.
As shown in the above table about constants of P– :
a = +1, −1, -2, +3, −4, +2, +1, -5, +7, −6...
This is the Perrin sequence for negative numbers from the second list item.
B = +0, +2, +3, +2, +5, +5, +7, +10, +12, +17...
This sequence is the Perrin sequence from P2.

2.3 P+ AND P− SEQUENCES: GENERALIZATION

It is obvious that the multiplication of the P+ and P− constants with the same degrees results in 1.
The general pattern of the variables describe these sequences:

	x^2	+x
P+	b	a * −1

P−	a	b

3. FIBONACCI SEQUENCES AND LUCAS SERIES

Adapting the notation used for Padovan and Fibonacci sequences,
F+ and F− means shifted Fibonacci sequences (e.g. F+ with degree = 1: F2/F1; F3/F2; F4/F3…). The shifted Lucas sequences will not be discussed separately since the values of their constants for any degree is equal to the values of the constants of the shifted Fibonacci sequences.

3.1 F+ SEQUENCES

degree	***phi^{n}***		***c***		***d***			***constant***
0	phi^{0}							1.0
1	phi^{1}	=	***0***	+	***1***	*	phi	1.618033988749895 (phi)
2	phi^{2}	=	***1***	+	***1***	*	phi	2.618033988749895
3	phi^{3}	=	***1***	+	***2***	*	phi	4.23606797749979
4	phi^{4}	=	***2***	+	***3***	*	phi	6.854101966249685
5	phi^{5}	=	***3***	+	***5***	*	phi	11.090169943749475
6	phi^{6}	=	***5***	+	***8***	*	phi	17.94427190999916
7	phi^{7}	=	***8***	+	***13***	*	phi	29.034441853748632
8	phi^{8}	=	***13***	+	***21***	*	phi	46.97871376374779
9	phi^{9}	=	***21***	+	***34***	*	phi	76.01315561749642
10	phi^{10}	=	***34***	+	***55***	*	phi	22.99186938124421

…

The constant is equal to **phi^{degree}** (this is true for the F− sequences, see below).
Another form to describe the constants:
c = **0, +1,+1,+2,+3,+5,+8,+13,+21,+34...**
This is the Fibonacci sequence from F1.
D = **+1,+1,+2,+3,+5,+8,+13,+21,+34,+55...**
This is the Fibonacci sequence from F2.

3.2 F− SEQUENCES

degree	***phi^{n}***			***c***		***d***			***constant***
0	phi^{0}							1.0	
−1	phi^{-1}	=	1/	(**0**	+	**1**	*	phi)	0.6180339887498949
−2	phi^{-2}	=	1/	(**1**	+	**1**	*	phi)	0.38196601125010515
−3	phi^{-3}	=	1/	(**1**	+	**2**	*	phi)	0.2360679774997897
−4	phi^{-4}	=	1/	(**2**	+	**3**	*	phi)	0.14589803375031546
−5	phi^{-5}	=	1/	(**3**	+	**5**	*	phi)	0.09016994374947424
−6	phi^{-6}	=	1/	(**5**	+	**8**	*	phi)	0.05572809000084122
−7	phi^{-7}	=	1/	(**8**	+	**13**	*	phi)	0.034441853748633025
−8	phi^{-8}	=	1/	(**13**	+	**21**	*	phi)	0.02128623625220819
−9	phi^{-9}	=	1/	(**21**	+	**34**	*	phi)	0.013155617496424838
−10	phi^{-10}	=	1/	(**34**	+	**55**	*	phi)	0.008130618755783348

…
As it was mentioned earlier, the sequences of F– = sequences of 1/F+.

4. CONCLUSION

It is not a surprise that these shifted sequences result in constants as in the case of phi or plastic number – the nature of these shifted sequences is the same as the nature of the original Fibonacci, Perrin etc. sequences.
On the other hand, it is remarkable that the strict rules to regulate the values of the constants of the shifted sequences based elusively on the Perrin or the Fibonacci sequences.

	Type of formula formula	**Variable numerical components (a/c) from**	**Variable numerical components (b/d) from**
P+	$x^3 - a * x^2 + b * x - 1$	P2(=a)	−P2 * −1 (=b)
P−	$x^3 - a * x^2 + b * x - 1$	−P2 (=a)	P2 (=b)
F+	c+d*phi	F1 (=c)	F2 (=d)
	phi^{degree}		
F−	1/(c+d*phi)	F1 (=c)	F2 (=d)
	phi^{degree}		

SOURCES

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Besides the programs written by me, Wolfram Alpha and mathway.com were used for some numerical calculations.

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